Ratio Cum Exponential In Regression-Type Mean Estimator Incorporating Empirical Distribution Function as Dual Application of Supplementary Variable Under Stratified Random Sampling Design

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ABSTRACT

In survey sampling for valid inferences, it depends on precise estimation of finite population parameters, such as the population mean. In this study, we present Ratio Cum Exponential In Regression-Type estimator under stratified random sampling design using empirical distribution function (EDF) as a dual of auxiliary variable. The bias and Mean Square Error (MSE) of the proposed estimators are derived up to first-order approximation. The proposed estimator has the minimum MSE and higher Percentage Relative Efficiency (PRE) from all the estimators which are considered as counterpart. In stratified random sampling, the dual use of auxiliary variable is more important when limited auxiliary information is available.

Keywords: Ratio Cum Exponential in Regression, Auxiliary variable, EDF, Stratified random sampling, MSE and PRE.

1 INTRODUCTION

Accurate estimation of population parameters, such as the mean, is a fundamental objective in survey sampling, as it forms the basis for informed decision-making and policy development. Estimators that efficiently capture the characteristics of the population are highly valued, particularly in scenarios where resources, time, and costs are constrained. To enhance the accuracy and efficiency of these estimators, researchers often leverage supplementary characteristics that are correlated with the variable of interest.

Supplementary characteristics are additional pieces of information that are readily available and exhibit a strong relationship with the study variable. Their utilization not only improves the precision of estimators but also reduces sampling variance, thereby leading to cost-effective and reliable population estimates. In stratified sampling designs, where the population is divided into homogeneous subgroups (strata), the integration of supplementary characteristics further refines the estimation process by accounting for within-strata variations and ensuring a more representative sample.

Over the years, researchers have proposed and analyzed various estimators, including ratio, regression, and exponential-type estimators, to make effective use of supplementary characteristics. These estimators exploit the relationship between the study variable and the supplementary characteristic to improve the efficiency of population mean estimation. Consequently, such methodologies have become an integral part.

2 Notations And Symbols

Assume that $S = \{S_1, S_2, \ldots, S_N\}$ is shows the population of N certain units that are divided into M strata, and j-th is size of stratum N_j for $j = 1, 2, \ldots, M$, provided that:

$$\sum_{j=1}^{M} N_j = N$$

Let Y and X be the study and auxiliary variables, which take values y_{hi} and x_{hi}, respectively, where $i = 1, 2, \ldots, N_j$ and $j = 1, 2, \ldots, M$. Assume that (SRSWOR) is used to extract a sample of size n_j from the j-th stratum in order to estimate the population mean.

The total sample size is denoted by:

$$\sum_{j=1}^{M} n_j = n$$

The total sample size is denoted by n. Take into account Y as the research variable, X as the auxiliary variable, and F_x as the auxiliary variable's EDF, respectively. The following provides the population means for the h-th stratum:

$$\overline{Y}_j = \frac{1}{N_j}\sum_{i=1}^{N_j} Y_{ij}, \quad \overline{X}_j = \frac{1}{N_j}\sum_{i=1}^{N_j} X_{ij}, \quad \overline{F_{x_j}} = \frac{1}{N_j}\sum_{i=1}^{N_j} F_{x_{ih}}$$

where Y_{ij}, X_{ij}, and $F_{x_{ij}}$ are the values of Y, X, and F_x for the i-th unit in the j-th stratum. The population means of Y, X, and F_x based on stratified random sampling are:

$$\overline{Y}_{st} = \overline{Y} = \sum_{j=1}^{M} W_j \overline{Y}_j, \quad \overline{X}_{st} = \overline{X} = \sum_{j=1}^{M} W_j \overline{X}_j, \quad \overline{F}_{st} = \overline{F_x} = \sum_{j=1}^{M} W_j \overline{F_{x_j}}$$

where $W_j = \frac{N_j}{N}$ is the stratum weight.
The sample means for the j-th stratum are:

$$\hat{Y}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} Y_{ij}, \quad \hat{X}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} X_{ij}, \quad \hat{F_{x_j}} = \frac{1}{n_j}\sum_{i=1}^{n_j} F_{x_{ij}}$$

Finally, the sample means of Y, X, and F_x based on stratified random sampling are:

$$\hat{Y}_{st} = \hat{Y} = \sum_{j=1}^{M} W_j \hat{Y}_j, \quad \hat{X}_{st} = \hat{X} = \sum_{j=1}^{M} W_j \hat{X}_j, \quad \hat{F}_{xst} = \hat{F}_x = \sum_{j=1}^{M} W_j \hat{F}_{x_j}$$

The population variances of Y, X, and F_x for the j-th stratum are given by:

$$S_{Yj}^2 = \frac{1}{N_j - 1} \sum_{i=1}^{N_j} (Y_{ij} - \overline{Y}_j)^2, \quad S_{Xj}^2 = \frac{1}{N_j - 1} \sum_{i=1}^{N_j} (X_{ij} - \overline{X}_j)^2, \quad S_{F_{x_j}}^2 = \frac{1}{N_j - 1} \sum_{i=1}^{N_j} (F_{x_{ij}} - \overline{F_{x_j}})^2$$

The population coefficients of variation for Y, X, and F_x for the j-th stratum are:

$$C_{Yj} = \frac{S_{Yj}}{\overline{Y}_j}, \quad C_{Xj} = \frac{S_{Xj}}{\overline{X}_j}, \quad C_{F_{x_j}} = \frac{S_{F_{x_j}}}{\overline{F_{x_j}}}$$

The covariance between Y, X, and F_x for the h-th stratum are given by:

$$S_{YXj} = \frac{1}{N_j - 1} \sum_{i=1}^{N_j} (Y_{ij} - \overline{Y}_j)(X_{ij} - \overline{X}_j), \quad S_{YF_{x_j}} = \frac{1}{N_j - 1} \sum_{i=1}^{N_j} (Y_{ij} - \overline{Y}_j)(F_{x_ij} - \overline{F_{x_j}}), \quad S_{XF_{x_j}} = \frac{1}{N_j - 1} \sum_{i=1}^{N_j} (X_{ij} -$$

The population correlation coefficients between (Y, X), (Y, F_x), and (X, F_x) for the h-th stratum are:

$$R_{YXj} = \frac{S_{YXj}}{S_{Yj} S_{Xj}}, \quad R_{YF_{x_j}} = \frac{S_{YF_{x_j}}}{S_{Yj} S_{F_{x_j}}}, \quad R_{XF_{x_j}} = \frac{S_{XF_{x_j}}}{S_{Xj} S_{F_{x_j}}},$$

$$R_{YX} = \frac{\sum_{j=1}^{M} W_j^2 \lambda_j R_{YXj} S_{Yj} S_{Xj}}{\sqrt{\sum_{j=1}^{M} W_j^2 \lambda_j S_{Yj}^2} \sqrt{\sum_{j=1}^{M} W_j^2 \lambda_j S_{Xj}^2}},$$

$$R_{yf_x} = \frac{\sum_{j=1}^{M} W_j^2 \lambda_j R_{yf_{x_j}} S_{yj} S_{f_{x-j}}}{\sqrt{\sum_{j=1}^{M} W_j^2 \lambda_j S_{yj}^2} \cdot \sqrt{\sum_{j=1}^{M} W_j^2 \lambda_j S_{f_{x_j}}^2}},$$

$$R_{xf_x} = \frac{\sum_{j=1}^{M} W_j^2 \lambda_j R_{xf_{x_j}} S_{x_j} S_{f_{x_j}}}{\sqrt{\sum_{j=1}^{M} W_j^2 \lambda_j S_{x_j}^2} \cdot \sqrt{\sum_{j=1}^{M} W_j^2 \lambda_j S_{f_{x_j}}^2}},$$

And

$$R_{y.xf_x}^2 = \frac{R_{yx}^2 + R_{yf_x}^2 - 2 R_{yx} R_{yf_x} R_{xf_x}}{1 - R_{xf_x}^2}$$

Relative discrepancies formulated for stratified random sampling
$e_1 = (\frac{\hat{\hat{Y}}}{\overline{Y}} - 1)$, $e_2 = (\frac{\hat{\hat{X}}}{\overline{X}} - 1)$, $e_3 = (\frac{\hat{\hat{R}}_x}{\overline{R}_x} - 1)$, $e_4 = (\frac{\hat{\hat{F}}_x}{\overline{F}_x} - 1)$, such that $E(e_i) = 0$ for i = 1, 2, 3, 4.
Now the Expected values are

$$E(e_1^2) = \sum_{j=1}^{M} W_j^2 \lambda_j^2 C_{y_j}^2 = D_{200},$$

$$E(e_2^2) = \sum_{j=1}^{M} W_j^2 \lambda_j^2 C_{x_j}^2 = D_{020},$$

$$E(e_3^2) = \sum_{j=1}^{M} W_j^2 \lambda_j^2 C_{r_{x_j}}^2 = D_{002},$$

$$E(e_4^2) = \sum_{j=1}^{M} W_j^2 \lambda_j^2 C_{f_{x_j}}^2 = D_{002},$$

$$E(e_1 e_2) = \sum_{j=1}^{M} W_j^2 \lambda_j^2 C_{y_j} C_{x_j} = D_{110},$$

$$E(e_1 e_3) = \sum_{j=1}^{M} W_h^2 \lambda_j^2 C_{y_j} C_{f_{x_j}} = D_{101}$$

And

$$E(e_2 e_3) = \sum_{hj1}^{M} W_j^2 \lambda_j^2 C_{x_j} C_{f_{x_j}} = D_{011}$$

3 Considered Available Estimators

In this portion we elaborate fave poplar available estimators in stratified random sampling design for the comparison to the proposed estimators. The bias, variance and (MSE) of the considered estimators are driven upto the first-order approximation. The following estimators are considered as counter part of the proposed estimators.

1. The simple estimator of $\bar{Y}$ under stratified random sampling is

$$\hat{Y}_{\text{SRSst}} = \frac{1}{n_1} \sum_{i=1}^{n_1} Y_i = \bar{Y} \tag{3.1}$$

$$\text{Var}\left(\hat{Y}_{\text{SRSst}}\right) = \hat{Y}^2 D_{200} \tag{3.2}$$

2. The Ratio estimator of $\bar{Y}$ under stratified random sampling given by Chocran [1940] is

$$\hat{\hat{Y}}_{\text{Rst}} = \hat{\hat{Y}} \left(\frac{\bar{X}}{\hat{\hat{X}}_{\text{st}}} \right) \tag{3.3}$$

$$\text{Bias}\left(\hat{\hat{Y}}_{\text{Rst}}\right) = \bar{Y}\left(D_{020} - D_{110}\right)$$

$$\text{MSE}\left(\hat{Y}_{\text{Rst}}\right) = \hat{Y}^2 \left(D_{020} - 2D_{110} + D_{200}\right) \tag{3.4}$$

3. The Product estimator of $\bar{Y}$ under stratified random sampling given by Murthy [1964] is

$$\hat{\bar{Y}}_{\text{Pst}} = \hat{\bar{Y}} \left(\frac{\bar{X}}{\hat{\bar{X}}_{\text{st}}} \right) \tag{3.5}$$

$$\text{Bias}\left(\hat{\bar{Y}}_{\text{Pst}}\right) = \bar{Y} D_{110}$$

$$\text{MSE}\left(\hat{\bar{Y}}_{\text{Pst}}\right) = \bar{Y}^2 D_{110} \tag{3.6}$$

4. The Regression estimator of $\bar{Y}$ under stratified random sampling is given as

$$\hat{\bar{Y}}_{\text{Regst}} = \hat{\bar{Y}}_{\text{st}} + p\left(\hat{\bar{X}} - \hat{\bar{X}}_{\text{st}}\right) \tag{3.7}$$

The $MSE_{(min.)}$ at optimal p is

$$\text{MSE}_{\text{min}}\left(\hat{Y}_{\text{Regst}}\right) = \bar{Y}^2 \frac{D_{200}D_{020} - D_{110}^2}{D_{020}} \tag{3.8}$$

after more simplification the Equ.(6.8) becomes as

$$\text{MSE}_{\text{min}}\left(\hat{Y}_{\text{Regst}}\right) = \bar{Y}^2 D_{200}\left(1 - R_{yx}^2\right) \tag{3.9}$$

5. The available difference type estimator in stratified random sampling are presented as:

$$\hat{\bar{Y}}_{\text{Dst}} = p_1 \hat{\bar{Y}}_{\text{st}} + p_2\left(\hat{\bar{X}} - \hat{\bar{X}}_{\text{st}}\right) \tag{3.10}$$

$$\text{Bias}\left(\hat{\bar{Y}}_{\text{Dst}}\right) = \bar{Y}\left(p_1 - 1\right)$$

$$\text{MSE}\left(\hat{\bar{Y}}_{\text{Dst}}\right) = \bar{Y}^2\left(p_1^2 D_{200} + p_2^2 D_{020} - 2p_1p_2D_{110}\right) \tag{3.11}$$

The optimum values of p_1 and p_2 are

$$p_{1\text{opt}} = \frac{D_{020}}{D_{020}D_{200} - D_{110}^2 + D_{020}}$$

$$p_{2\text{opt}} = \frac{\bar{Y} D_{110}}{\bar{X}\left(D_{200}D_{020} - D_{110}^2 + D_{020}\right)}$$

$MSE_{(min.)}$ of the $\bar{Y}_{Dst}$ at the optimal values of p_1 and p_2 is

$$\text{MSE}_{\text{min}}\left(\hat{\bar{Y}}_{\text{Dst}}\right) = \bar{Y}^2 \frac{D_{200}D_{020} - D_{110}^2}{D_{200}D_{020} - D_{110}^2 + D_{020}} \tag{3.12}$$

After simplification the $MSE_{(min.)}$ of $\bar{Y}_{Dst}$ is

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Dst}}\right)=\frac{\bar{Y}^2 D_{200}\left(1-R_{yx}^2\right)}{\left(1+D_{200}\left(1-R_{yx}^2\right)\right)} \tag{3.13}$$

6. The Exponential estimator given by Grover et al.[2014] is

$$\hat{\hat{Y}}_{\mathrm{Est}}=\hat{\hat{Y}}_{\mathrm{st}}\exp\left(\frac{a_1\left(\hat{\hat{X}}-\hat{X}_{\mathrm{st}}\right)}{a_1\left(\hat{X}+\hat{X}_{\mathrm{st}}\right)+2a_2}\right) \tag{3.14}$$

$$\mathrm{Bias}\left(\hat{\hat{Y}}_{\mathrm{Est}}\right)=\bar{Y}\left(\frac{3}{8}\theta^2 D_{020}-\frac{1}{2}\theta D_{110}\right)$$

$$\mathrm{MSE}\left(\hat{\hat{Y}}_{\mathrm{Est}}\right)=\bar{Y}^2\left(\frac{1}{4}\left(4D_{200}+\theta^2 D_{020}-4\theta D_{110}\right)\right) \tag{3.15}$$

where theta is

$$\theta=\frac{a_1\bar{X}}{\left(a_1\bar{X}+a_2\right)}.$$

7. Kaur et al.[2018] regression cum exponential-type estimator is

$$\hat{\hat{Y}}_{\mathrm{GKst}}=p_3\hat{\hat{Y}}_{\mathrm{st}}+p_4\left(\hat{\hat{X}}-\hat{\hat{X}}_{\mathrm{st}}\right)\exp\left(\frac{K\left(\hat{\hat{X}}-\hat{\hat{X}}_{\mathrm{st}}\right)}{K\left(\hat{\hat{X}}+\hat{\hat{X}}_{\mathrm{st}}\right)+2L}\right) \tag{3.16}$$

Bias of the estimator $\hat{\hat{Y}}_{\mathrm{GKst}}$ is

$$\mathrm{Bias}\left(\hat{\hat{Y}}_{\mathrm{GKst}}\right)=\bar{Y}\left(p_3-1\right)$$

MSE of the estimator $\hat{\hat{Y}}_{\mathrm{GKst}}$ is

$$\begin{aligned}\mathrm{MSE}\left(\hat{\hat{Y}}_{\mathrm{GKst}}\right)=&\,p_4^2\bar{X}^2 D_{020}+p_3^2\bar{Y}^2 D_{200}+2\theta p_3 p_4\bar{Y}\bar{X}D_{020}-2p_3p_4\bar{Y}\bar{X}D_{110}\\&+\bar{Y}^2\left(1-2p_3\right)+\theta p_3^2\bar{Y}^2+p_3\bar{Y}^2 D_{110}-\theta p_4\bar{Y}\bar{X}D_{020}\\&-2\theta p_3^2\bar{Y}^2 D_{110}-\frac{3}{4}\theta^2 p_3\bar{Y}^2 D_{020}+\theta^2 p_3^2\bar{Y}^2 D_{020}\end{aligned} \tag{3.17}$$

The optimum values of p_3 and p_4 are

$$p_{3\text{opt}} = \frac{D_{020}\theta^2}{8 - D_{200}D_{020} + D_{110}^2 - D_{020}}$$

and

$$p_{4\text{opt}} = \frac{\bar{Y}\theta^3 \left(D_{020}^2 - \theta^2 D_{020}D_{110} + 4\theta D_{200}D_{020} - 4\theta D_{110}^2 - 4\theta D_{020} + 8D_{110}\right)}{8\bar{X}\left(D_{200}D_{020} - D_{110}^2 + D_{020}\right)}$$

Minimum MSE of the estimator $\hat{\bar{Y}}_{\text{GKst}}$ for the optimal values of p_3 and p_4 is

$$\text{MSE}_{\min}\left(\hat{\bar{Y}}_{\text{GKst}}\right) = \frac{\bar{Y}^2}{64}\left(64 - 16\theta^2 D_{020} - \frac{\left(D_{020} - 8 + \theta^2 D_{020}\right)^2}{D_{020}\left(1 + D_{200}\right) - D_{110}^2}\right) \tag{3.18}$$

or

$$\text{MSE}_{\min}\left(\hat{\bar{Y}}_{\text{GKst}}\right) = \text{MSE}_{\min}\left(\hat{\bar{Y}}_{\text{Regst}}\right) - \frac{\bar{Y}^2\theta^2\left(D_{020}^2 - 8D_{110}^2 + 8D_{020}D_{200}\right)^2}{64D_{020}^2\left(1 + D_{200}\left(1 - R_{yx}^2\right)\right)} \tag{3.19}$$

which show that $\hat{\bar{Y}}_{\text{GKst}}$ is more efficient then

4 Proposed Ratio Cum Exponential In Regression-Type Mean Estimator

The use of supplementary characteristics for estimating population parameters, such as the mean, can enhance the efficiency of estimators. In such cases, a characteristic that is correlated with the study characteristic is considered. However, there are situations where a correlated characteristic is either unavailable or limited to just one. It is commonly observed that when a characteristic is correlated with the study variable, its Empirical Distribution Function (EDF) also exhibits a correlation with the study variable. This EDF can be treated as a new characteristic and utilized to further reduce the Mean Square Error (MSE) of the estimator. In the proposed regression-cum-exponential type family of estimators, both the supplementary characteristic and its EDF are employed as dual supplementary characteristics under stratified random sampling. The bias and MSE of the proposed regression-cum-exponential type estimator are derived up to the first-order approximation.

Proposed Estimator $\hat{\bar{Y}}_{\text{Pst}}$ with unknown p_5, p_6, and p_7

$$\hat{\bar{Y}}_{\text{Pst}} = \left[p_5\hat{\bar{Y}}_{\text{st}} + p_6\left(\bar{X} - \hat{\bar{X}}_{\text{st}}\right) + p_7\left(\bar{F}_x - \hat{\bar{F}}_{\text{xst}}\right)\right]\exp\left(\frac{a_1\left(\bar{X} - \hat{\bar{X}}_{\text{st}}\right)}{a_1\left(\bar{X} + \hat{\bar{X}}_{\text{st}}\right) + 2a_2}\right) \tag{4.1}$$

$$\hat{\bar{Y}}_{\text{Pst}} = \{p_5\tilde{Y}(1 + e_1) - p_6 e_2 - p_7 e_3\}\left(1 - \frac{1}{2}\theta_2 e_2 + \frac{3}{8}\theta_2^2 e_2^2 + \cdots\right). \tag{24}$$

After simplification, we have:

$$\left(\hat{\hat{Y}}_{\text{Pst}} - \tilde{Y}\right) = -\tilde{Y} + p_5\tilde{Y}e_1 - \frac{1}{2}p_5\tilde{Y}e_2 - p_6e_2 - p_7e_3 + \frac{3}{8}\theta_2 p_5\tilde{Y}e_1^2 + \frac{1}{2}\theta_2 p_5\tilde{Y}e_1e_2$$
$$-\frac{1}{2}p_5\tilde{Y}e_1e_2 \; -\frac{1}{2}p_6e_1e_2 + \frac{1}{2}p_7e_2e_3.$$

Derivations of Bias and MSE of the $\hat{\hat{Y}}_{\text{Pst}}$ are,

$$B(\hat{\hat{Y}}_{\text{Pst}}) = \tilde{Y}(Q_5-1) + \frac{3}{8}\theta^2 Q_5\tilde{Y}V_{020} + \frac{1}{2}Q_6V_{020} - \frac{1}{2}\theta Q_5\tilde{Y}V_{110} + \frac{1}{2}\theta Q_7V_{011} \tag{4.2}$$

$$\begin{aligned} MSE(\hat{\hat{Y}}_{\text{Pst}}) &= \bar{Y}^2(p_5-1)^2 + p_5^2\bar{Y}^2D_{200} + p_6^2D_{020} + p_7^2D_{002} + \theta^2p_5^2\bar{Y}^2D_{020} \\ &-p_5p_6\bar{Y}^2D_{020} + 2\theta p_5^2\bar{Y}D_{020} - \frac{3}{4}\theta^2p_5^2\bar{Y}^2D_{110} + 2\theta p_5^2\bar{Y}^2D_{110} \\ &-2p_5p_6\bar{Y}D_{110} - 2p_5p_7\bar{Y}D_{011} + 2\theta p_5p_7\bar{Y}D_{011} - 2p_6p_7D_{011} \end{aligned} \tag{4.3}$$

The values of p_5, p_6, and p_7, as optimum level are,

$$p_{5(\text{opt})} = \frac{8-\theta^2D_{020}}{8\left[1+D_{200}\left(1-R^2_{yxf_x}\right)\right]}$$

$$p_{6(\text{opt})} = \Big[\theta^3D_{020}^{1/2}(R^2_{xf_x}-1) + D_{200}^{1/2}(-8+\theta^2D_{020})(R_{yx}-R_{xf_x}R_{yf_x})$$
$$+4\theta D_{020}^{1/2}(R^2_{xf_x}-1) - 1 + D_{200}(1-R^2_{yxf_x})\Big/ 8D_{020}^{1/2}(R^2_{xf_x}-1)[-1+D_{200}(1-R^2_{yxf_x})$$

$$p_{7(\text{opt})} = \frac{\bar{Y}D_{020}^{1/2}D_{200}(8-\theta^2D_{020})(R_{yx}-R_{xf_x}R_{yf_x})}{8D_{020}^{1/2}(R^2_{xf_x}-1)[-1+D_{200}(1-R^2_{yxf_x})]}$$

After putting these optimal level of p_5, p_6, and p_7 than the $MSE_{(min.)}$ of the proposed estimator as

$$MSE_{min}(\hat{\hat{Y}}_{\text{Pst}}) = \frac{\bar{Y}^2\left[64D_{200}\left(1-R^2_{y.xf_x}-\theta^4D^2_{020}-16\theta^2D_{020}D_{200}\right)\left(1-R^2_{y.xf_x}\right)\right]}{64\left[1+D_{200}\left(1-R^2_{y.xf_x}\right)\right]} \tag{4.4}$$

where

$$R^2_{y.xf_x} = \frac{D^2_{110}D_{002} + D^2_{101}D_{020} - 2D_{101}D_{110}D_{011}}{D_{200}(D_{020}D_{002}-D^2_{011})}.$$

then equation (6.23) may be written as

$$\text{MSE}_{\min}\hat{\hat{Y}}_{\text{Pst}} = \text{MSE}_{\min}\hat{Y}_{\text{Regst}} - Q_1 - Q_2, \tag{4.5}$$

where

$$Q_1 = \frac{\bar{Y}^2\left(\theta^2D^2_{020} - 8D^2_{110} + 8D_{020}D_{200}\right)^2}{64D^2_{020}\left(1+D_{200}\left(1-R^2_{yx}\right)\right)},$$

$$Q_2 = \frac{\bar{Y}^2\left(\theta^2D_{020}-8\right)^2\left(D_{020}D_{101}-D_{011}D_{110}\right)^2}{64D^2_{020}D_{002}\left(1-R^2_{xf_x}\right)\left(1+D_{200}\left(1-R^2_{yx}\right)\right)\left(1+D_{200}\left(1-R^2_{y.xf_x}\right)\right)}.$$

5 Efficiency Comparisons

In this portion we compared all the considered estimators efficiency with the proposed estimator as given bleow.

(i) By taking (6.24) and (6.2),

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) < \mathrm{Var}\left(\hat{\hat{Y}}_{\mathrm{SRS}}\right) \text{ if } \hat{Y}^2 V_{200}^2 R_{yx}^2 + Q_1 + Q_2 > 0. \tag{5.1}$$

(ii) By taking (6.24) and (6.4),

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) < \mathrm{MSE}\left(\hat{\hat{Y}}_{\mathrm{Rst}}\right) \text{ if } \frac{1}{D_{020}}\left(D_{020} - D_{110}\right)^2 + Q_1 + Q_2 > 0. \tag{5.2}$$

(iii) By taking (4.8) and (3.6),

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) < \mathrm{MSE}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) \text{ if } \frac{1}{D_{020}}\left(D_{020} + D_{110}\right)^2 + Q_1 + Q_2 > 0. \tag{5.3}$$

(iv) By taking (6.24) and (3.9),

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) < \mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Regst}}\right) \text{ if } Q_1 + Q_2 > 0. \tag{5.4}$$

(v) By taking (6.24) and (6.11),

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) < \mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{RDst}}\right) \text{ if } \frac{\bar{Y}^2\theta^2 D_{020}\left(\theta^2 D_{020} + 16 D_{200}\left(1 - R_{yx}^2\right)\right)}{64\left[1 + \left(1 - R_{yx}^2\right)\right]} + Q_2 > 0. \tag{5.5}$$

(vi) By taking (6.13) and (6.24),

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) < \mathrm{MSE}\left(\hat{\hat{Y}}_{\mathrm{Sst}}\right) \text{ if } \frac{1}{D_{020}}\left(\theta D_{020}^2 - D_{110}\right)^2 + Q_1 + Q_2 > 0. \tag{5.6}$$

(vii) By taking (6.18) and (6.24),

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{Pst}}\right) < \mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{\mathrm{GKst}}\right) \text{ if } Q_2 > 0. \tag{5.7}$$

6 Empirical and Simulation Study

6.1 Empirical Study

In this analysis, the mathematical results are demonstrated to assess the efficiency of all estimators. Four different data sets are utilized for this evaluation. The results obtained from these data sets are detailed in Tables 6.2–6.5. The percentage relative efficiency of the estimator $\hat{Y}_i$ in comparison to $\hat{\hat{Y}}_{\text{Pst}}$ is computed as follows:

$$\text{PRE}\left(\hat{\hat{Y}}_i, \hat{\hat{Y}}_{\text{Pst}}\right) = \frac{\text{Var}\left(\hat{\hat{Y}}_{\text{Pst}}\right)}{\text{MSE}_{(min.)}\left(\hat{\hat{Y}}_i\right)} \times 100, \tag{6.1}$$

where $i = SRSst, Rst, \ldots, GKst$.

The percentage relative efficiency(PRE %) of the four data sets, is listed in Tables 6.6–6.9.

Set 1 (source: Koyuncu and Kadilar [18]):

Y : The number of instructors and

X: The number of trainees in 2007 for 923 districts in six regions.

Set 2 (source: Kadilar and Cingi [19]):

Y: The yield of apples in 1999 and

X: The yield of apples in 1998.

6.2 Simulation Study

A simulation study is conducted to evaluate the efficiency of the proposed estimators for the stratified sampling method. This is achieved using information from a single supplementary characteristic X and an adjusted rank set sampling approach applied to the supplementary characteristic X, as well as to the constructed variable F_x.

Table 1: Overview of the Data for Set 1 (Case I)

h	**Nh**	**nh**	**wh**	θ_h	$\bar{Y}_h$	$\bar{X}_h$	$\bar{F}_h$	S_{yh}	S_{xh}	S_{fh}	R_{yxh}	R_{yfh}	R_{xfh}
1	127	31	0.1375	0.0244	704	20805	64	883.83	486.75	6.80	0.9366	0.8239	0.7834
2	117	21	0.1267	0.0390	413	9212	59	644.92	5180.77	3.92	0.9956	0.6584	0.6517
3	103	29	0.1115	0.0248	74	14309	52	1033.46	27549.7	9.87	0.9937	0.6337	0.6237
4	170	38	0.1841	0.0204	425	9479	86	810.58	8218.93	49.21	0.9834	0.6360	0.6442
5	205	22	0.2221	0.0406	267	5570	103	403.65	8497.77	59.32	0.9893	0.6595	0.6655
6	201	39	0.2177	0.0207	394	12998	101	711.72	3094.14	58.16	0.9651	0.5863	0.6162

Table 2: Overview of the Data for Set 1 (Case II)

h	**Nh**	**nh**	**wh**	θ_h	$\bar{Y}_h$	$\bar{X}_h$	$\bar{F}_h$	S_{yh}	S_{xh}	S_{fh}	R_{yxh}	R_{yfh}	R_{xfh}
1	127	31	0.1375	0.0244	704	498	64	883.83	55.58	36.80	0.9366	0.8239	0.7834
2	117	21	0.1267	0.0391	413	318	59	644.92	65.45	33.92	0.9956	0.6584	0.6517
3	103	29	0.1115	0.0248	574	431	52	1033.46	12.95	29.87	0.9937	0.6337	0.6237
4	170	38	0.1841	0.0204	425	311	86	810.58	458.02	49.22	0.9834	0.6360	0.6442
5	205	22	0.2221	0.0406	267	227	103	410.65	60.85	59.32	0.9893	0.6595	0.6655
6	201	39	0.2177	0.0207	394	314	101	711.72	97.05	58.16	0.9651	0.5863	0.6162

Table 3: Overview of the Data for Set 2 (Case I)

h	Nh	nh	wh	θ_h	$\bar{Y}_h$	$\bar{X}_h$	$\bar{F}_h$	S_{yh}	S_{xh}	S_{fh}	R_{yxh}	R_{yfh}	R_{xfh}
1	106	9	0.1241	0.1017	1537	24376	54	6425.08	49189.08	30.74	0.8156	0.3349	0.5930
2	106	17	0.1241	0.0494	2213	27422	54	11551.53	57460.61	30.74	0.8559	0.2816	0.6031
3	94	38	0.1100	0.0157	9384	72410	48	29907.48	160757.31	27.28	0.9011	0.4637	0.5873
4	171	67	0.2002	0.0090	5588	74365	87	28643.42	285603.12	49.51	0.9858	0.2981	0.3654
5	204	7	0.2389	0.4657	967	26442	103	2389.77	45402.78	45.41	0.7130	0.4547	0.6206
6	173	2	0.2026	0.4942	404	9844	87	945.74	18793.96	50.08	0.8935	0.5435	0.6262

Table 4: Overview of the Data for Set 2 (Case II)

h	Nh	nh	wh	θ_h	$\bar{Y}_h$	$\bar{X}_h$	$\bar{F}_h$	S_{yh}	S_{xh}	S_{fh}	R_{yxh}	R_{yfh}	R_{xfh}
1	106	9	0.1241	0.1017	1537	24712	54	6425.08	49134.76	30.74	0.8156	0.3346	0.5956
2	106	17	0.1241	0.0494	2213	26840	54	11551.53	53978.71	30.74	0.8359	0.2814	0.6246
3	94	38	0.1100	0.0157	9384	72722	48	29907.48	161109.50	27.27	0.8971	0.4626	0.5893
4	171	67	0.2002	0.0090	5588	73191	87	28643.42	262495.61	49.50	0.9814	0.2979	0.3885
5	204	7	0.2389	0.1379	967	26834	103	2389.77	45174.26	59.03	0.7107	0.4541	0.6317
6	173	2	0.2026	0.4942	404	9903	87	945.74	18977.28	50.08	0.8697	0.5366	0.6283

Table 5: MSEs for Set 1 (case I).

Estimator	MSE	Estimator	MSE	Estimator	MSE	Estimator	MSE
$\bar{Y}_{\text{SRSst}}$	2229.266	$\hat{Y}^{(1)}_{\text{Sst}}$	5096.365	$\hat{Y}^{(1)}_{\text{GKst}}$	192.9490	$\hat{Y}^{(1)}_{\text{Pst}}$	185.1848
$\bar{Y}_{\text{Rst}}$	9205.298	$\hat{Y}^{(2)}_{\text{Sst}}$	604.1097	$\hat{Y}^{(2)}_{\text{GKst}}$	192.9539	$\hat{Y}^{(2)}_{\text{Pst}}$	185.1897
$\bar{Y}_{\text{Pst}}$	216.4183	$\hat{Y}^{(3)}_{\text{Sst}}$	602.4522	$\hat{Y}^{(3)}_{\text{GKst}}$	192.9485	$\hat{Y}^{(3)}_{\text{Pst}}$	185.1844
$\bar{Y}_{\text{Regst}}$	194.2832	$\hat{Y}^{(4)}_{\text{Sst}}$	603.4212	$\hat{Y}^{(4)}_{\text{GKst}}$	192.9516	$\hat{Y}^{(4)}_{\text{Pst}}$	185.1875
$\bar{Y}_{\text{Dst}}$	194.0853	$\hat{Y}^{(5)}_{\text{Sst}}$	602.5302	$\hat{Y}^{(5)}_{\text{GKst}}$	192.9487	$\hat{Y}^{(5)}_{\text{Pst}}$	185.1846
		$\hat{Y}^{(6)}_{\text{Sst}}$	602.4947	$\hat{Y}^{(6)}_{\text{GKst}}$	192.9486	$\hat{Y}^{(6)}_{\text{Pst}}$	185.1845
		$\hat{Y}^{(7)}_{\text{Sst}}$	602.5978	$\hat{Y}^{(7)}_{\text{GKst}}$	192.9490	$\hat{Y}^{(7)}_{\text{Pst}}$	185.1849
		$\hat{Y}^{(8)}_{\text{Sst}}$	602.4488	$\hat{Y}^{(8)}_{\text{GKst}}$	192.9485	$\hat{Y}^{(8)}_{\text{Pst}}$	185.1844
		$\hat{Y}^{(9)}_{\text{Sst}}$	604.1488	$\hat{Y}^{(9)}_{\text{GKst}}$	192.9540	$\hat{Y}^{(9)}_{\text{Pst}}$	185.1898
		$\hat{Y}^{(10)}_{\text{Sst}}$	2226.835	$\hat{Y}^{(10)}_{\text{GKst}}$	194.0853	$\hat{Y}^{(10)}_{\text{Pst}}$	186.2958

Table 6: MSEs for Set 1 (case II).

Estimator	MSE	Estimator	MSE	Estimator	MSE	Estimator	MSE
$\bar{Y}_{\text{SRSst}}$	2229.266	$\hat{Y}^{(1)}_{\text{Sst}}$	4240.170	$\hat{Y}^{(1)}_{\text{GKst}}$	101.1275	$\hat{Y}^{(1)}_{\text{Pst}}$	79.35476
$\bar{Y}_{\text{Rst}}$	6936.636	$\hat{Y}^{(2)}_{\text{Sst}}$	914.5469	$\hat{Y}^{(2)}_{\text{GKst}}$	101.1566	$\hat{Y}^{(2)}_{\text{Pst}}$	79.38443
$\bar{Y}_{\text{Pst}}$	193.2885	$\hat{Y}^{(3)}_{\text{Sst}}$	877.6646	$\hat{Y}^{(3)}_{\text{GKst}}$	101.1242	$\hat{Y}^{(3)}_{\text{Pst}}$	79.35178
$\bar{Y}_{\text{Regst}}$	101.5021	$\hat{Y}^{(4)}_{\text{Sst}}$	907.0444	$\hat{Y}^{(4)}_{\text{GKst}}$	101.1503	$\hat{Y}^{(4)}_{\text{Pst}}$	79.35956
$\bar{Y}_{\text{Dst}}$	101.4481	$\hat{Y}^{(5)}_{\text{Sst}}$	880.3341	$\hat{Y}^{(5)}_{\text{GKst}}$	101.1267	$\hat{Y}^{(5)}_{\text{Pst}}$	79.35404
		$\hat{Y}^{(6)}_{\text{Sst}}$	897.7216	$\hat{Y}^{(6)}_{\text{GKst}}$	101.1276	$\hat{Y}^{(6)}_{\text{Pst}}$	79.35348
		$\hat{Y}^{(7)}_{\text{Sst}}$	881.2793	$\hat{Y}^{(7)}_{\text{GKst}}$	101.1276	$\hat{Y}^{(7)}_{\text{Pst}}$	79.35178
		$\hat{Y}^{(8)}_{\text{Sst}}$	877.5927	$\hat{Y}^{(8)}_{\text{GKst}}$	101.1242	$\hat{Y}^{(8)}_{\text{Pst}}$	79.35178
		$\hat{Y}^{(9)}_{\text{Sst}}$	2227.442	$\hat{Y}^{(9)}_{\text{GKst}}$	101.4481	$\hat{Y}^{(9)}_{\text{Pst}}$	79.63742

Table 7: MSEs for Set 2 (case I).

Estimator	MSE	Estimator	MSE	Estimator	MSE	Estimator	MSE
$\bar{Y}_{\text{SRSst}}$	6977.896	$\hat{Y}^{(1)}_{\text{Sst}}$	1225952	$\hat{Y}^{(1)}_{\text{GKst}}$	214502.1	$\hat{Y}^{(1)}_{\text{Prst}}$	197729.1
$\bar{Y}_{\text{Rst}}$	1949400	$\hat{Y}^{(2)}_{\text{Sst}}$	365071.7	$\hat{Y}^{(2)}_{\text{GKst}}$	214508.7	$\hat{Y}^{(2)}_{\text{Pst}}$	197283.5
$\bar{Y}_{\text{Pst}}$	220715.9	$\hat{Y}^{(3)}_{\text{Sst}}$	364798.4	$\hat{Y}^{(3)}_{\text{GKst}}$	214507.8	$\hat{Y}^{(3)}_{\text{Pst}}$	197278.7
$\bar{Y}_{\text{Regst}}$	222881.3	$\hat{Y}^{(4)}_{\text{Sst}}$	364915.7	$\hat{Y}^{(4)}_{\text{GKst}}$	214509.5	$\hat{Y}^{(4)}_{\text{Pst}}$	197278.5
$\bar{Y}_{\text{R,Dst}}$	217241.8	$\hat{Y}^{(5)}_{\text{Sst}}$	364803.5	$\hat{Y}^{(5)}_{\text{GKst}}$	214501.8	$\hat{Y}^{(5)}_{\text{Pst}}$	197283.4
		$\hat{Y}^{(6)}_{\text{Sst}}$	364800.4	$\hat{Y}^{(6)}_{\text{GKst}}$	214501.8	$\hat{Y}^{(6)}_{\text{Pst}}$	197283.4
		$\hat{Y}^{(10)}_{\text{Sst}}$	697286.1	$\hat{Y}^{(10)}_{\text{GKst}}$	217241.8	$\hat{Y}^{(10)}_{\text{Pst}}$	198921.3

Table 8: MSEs for Set 2 (case II).

Estimator	MSE	Estimator	MSE	Estimator	MSE	Estimator	MSE
$\bar{Y}_{\text{SRSst}}$	6977.896	$\hat{Y}^{(1)}_{\text{Sst}}$	1225952	$\hat{Y}^{(1)}_{\text{GKst}}$	228480.8	$\hat{Y}^{(1)}_{\text{Pst}}$	207876.6
$\bar{Y}_{\text{Rst}}$	1878241	$\hat{Y}^{(2)}_{\text{Sst}}$	365071.7	$\hat{Y}^{(2)}_{\text{GKst}}$	228483.4	$\hat{Y}^{(2)}_{\text{Pst}}$	207878.2
$\bar{Y}_{\text{Pst}}$	243316.9	$\hat{Y}^{(3)}_{\text{Sst}}$	364798.4	$\hat{Y}^{(3)}_{\text{GKst}}$	228484.0	$\hat{Y}^{(3)}_{\text{Pst}}$	207878.7
$\bar{Y}_{\text{Regst}}$	237552.8	$\hat{Y}^{(4)}_{\text{Sst}}$	364915.7	$\hat{Y}^{(4)}_{\text{GKst}}$	228483.3	$\hat{Y}^{(4)}_{\text{Pst}}$	207878.3
$\bar{Y}_{\text{Dst}}$	231157.0	$\hat{Y}^{(5)}_{\text{Sst}}$	364803.5	$\hat{Y}^{(5)}_{\text{GKst}}$	228480.8	$\hat{Y}^{(5)}_{\text{Pst}}$	207876.2
		$\hat{Y}^{(10)}_{\text{Sst}}$	697286.1	$\hat{Y}^{(10)}_{\text{GKst}}$	231157.0	$\hat{Y}^{(10)}_{\text{Pst}}$	210333.4

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Table 9: Results of a simulation showing the percentage relative efficiency of the proposed exponential cum ratio in regression-type estimator in comparison to the current estimators for various strata.

Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)
$\tilde{Y}_{SRSst}$	100.00	$\tilde{Y}_{S^{(1)}}$	145.79	$\tilde{Y}_{GK_{st1}}$	186.6	$\tilde{Y}_{Pst_1}$	1518.9
$\tilde{Y}_{R_{st}}$	156.74	$\tilde{Y}_{S^{(2)}}$	100.21	$\tilde{Y}_{GK_{st2}}$	208.9	$\tilde{Y}_{Pst_2}$	1718.9
$\tilde{Y}_{Pst}$	101.98	$\tilde{Y}_{S^{(3)}}$	170.96	$\tilde{Y}_{GK_{st3}}$	754.9	$\tilde{Y}_{Pst_3}$	1280.78
$\tilde{Y}_{Reg_{st}}$	145.62	$\tilde{Y}_{S^{(4)}}$	149.49	$\tilde{Y}_{GK_{st4}}$	416.10	$\tilde{Y}_{Pst_4}$	1281.55
$\tilde{Y}_{D_{st}}$	137.62	$\tilde{Y}_{S^{(5)}}$	110.21	$\tilde{Y}_{GK_{st5}}$	209.19	$\tilde{Y}_{Pst_5}$	1291.51
		$\tilde{Y}_{S^{(6)}}$	100.11	$\tilde{Y}_{GK_{st6}}$	210.19	$\tilde{Y}_{Pst_6}$	1254.59
		$\tilde{Y}_{S^{(7)}}$	179.00	$\tilde{Y}_{GK_{st7}}$	861.9	$\tilde{Y}_{Pst_7}$	1921.51
		$\tilde{Y}_{S^{(8)}}$	100.12	$\tilde{Y}_{GK_{st8}}$	254.9	$\tilde{Y}_{Pst_8}$	1890.59
		$\tilde{Y}_{S^{(9)}}$	162.35	$\tilde{Y}_{GK_{st9}}$	233.78	$\tilde{Y}_{Pst_9}$	1489.01
		$\tilde{Y}_{S^{(10)}}$	172.54	$\tilde{Y}_{GK_{st10}}$	243.58	$\tilde{Y}_{Pst_{10}}$	1220.45

Table 10: PRE(%) for Set 1 (case I)

Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)
$\tilde{Y}_{SRSst}$	100.00	$\tilde{Y}_{S^{(1)}}$	43.38	$\tilde{Y}_{GK_{st1}}$	1155.39	$\tilde{Y}_{Pst_1}$	1203.81
$\tilde{Y}_{R_{st}}$	25.20	$\tilde{Y}_{S^{(2)}}$	368.56	$\tilde{Y}_{GK_{st2}}$	1155.34	$\tilde{Y}_{Pst_2}$	1203.72
$\tilde{Y}_{Pst}$	1029.98	$\tilde{Y}_{S^{(3)}}$	370.21	$\tilde{Y}_{GK_{st3}}$	1155.37	$\tilde{Y}_{Pst_3}$	1203.78
$\tilde{Y}_{Reg_{st}}$	1145.62	$\tilde{Y}_{S^{(4)}}$	1203.83	$\tilde{Y}_{GK_{st4}}$	1155.33	$\tilde{Y}_{Pst_4}$	1203.91
$\tilde{Y}_{D_{st}}$	1146.79	$\tilde{Y}_{S^{(5)}}$	370.25	$\tilde{Y}_{GK_{st5}}$	1155.32	$\tilde{Y}_{Pst_5}$	1203.88
		$\tilde{Y}_{S^{(6)}}$	368.15	$\tilde{Y}_{GK_{st6}}$	1155.38	$\tilde{Y}_{Pst_6}$	1203.80
		$\tilde{Y}_{S^{(7)}}$	370.02	$\tilde{Y}_{GK_{st7}}$	1155.36	$\tilde{Y}_{Pst_7}$	1203.93
		$\tilde{Y}_{S^{(8)}}$	369.63	$\tilde{Y}_{GK_{st8}}$	1155.37	$\tilde{Y}_{Pst_8}$	1203.89
		$\tilde{Y}_{S^{(9)}}$	370.15	$\tilde{Y}_{GK_{st9}}$	1155.38	$\tilde{Y}_{Pst_9}$	1203.95
		$\tilde{Y}_{S^{(10)}}$	100.19	$\tilde{Y}_{GK_{st10}}$	1148.61	$\tilde{Y}_{Pst_{10}}$	1196.94

Table 11: PRE(%) for Set 1 (case II)

Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)
$\tilde{Y}_{SRSst}$	100.00	$\tilde{Y}_{S^{(1)}}$	32.38	$\tilde{Y}_{GK_{st1}}$	1155.39	$\tilde{Y}_{Pst_1}$	1203.81
$\tilde{Y}_{R_{st}}$	32.20	$\tilde{Y}_{S^{(2)}}$	368.56	$\tilde{Y}_{GK_{st2}}$	1153.34	$\tilde{Y}_{Pst_2}$	1203.72
$\tilde{Y}_{Pst}$	1153.98	$\tilde{Y}_{S^{(3)}}$	370.21	$\tilde{Y}_{GK_{st3}}$	1155.37	$\tilde{Y}_{Pst_3}$	1203.78
$\tilde{Y}_{Reg_{st}}$	2195.62	$\tilde{Y}_{S^{(4)}}$	1203.83	$\tilde{Y}_{GK_{st4}}$	1155.33	$\tilde{Y}_{Pst_4}$	1203.91
$\tilde{Y}_{D_{st}}$	2197.79	$\tilde{Y}_{S^{(5)}}$	370.25	$\tilde{Y}_{GK_{st5}}$	1155.32	$\tilde{Y}_{Pst_5}$	1203.88
		$\tilde{Y}_{S^{(6)}}$	368.15	$\tilde{Y}_{GK_{st6}}$	1155.38	$\tilde{Y}_{Pst_6}$	1203.80
		$\tilde{Y}_{S^{(7)}}$	370.02	$\tilde{Y}_{GK_{st7}}$	1155.36	$\tilde{Y}_{Pst_7}$	1203.93
		$\tilde{Y}_{S^{(8)}}$	369.63	$\tilde{Y}_{GK_{st8}}$	1155.37	$\tilde{Y}_{Pst_8}$	1203.89
		$\tilde{Y}_{S^{(9)}}$	370.15	$\tilde{Y}_{GK_{st9}}$	1155.38	$\tilde{Y}_{Pst_9}$	1203.95
		$\tilde{Y}_{S^{(10)}}$	100.19	$\tilde{Y}_{GK_{st10}}$	1148.61	$\tilde{Y}_{Pst_{10}}$	1196.94

Table 12: PRE(%) for Set 2 (case I)

Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)
$\hat{Y}_{S_R}^{(1)}$	100.00	$\hat{Y}_{S_R}^{(2)}$	56.92	$\hat{Y}_{S_R}^{(3)}$	325.31
$\hat{Y}_R$	35.80	$\hat{Y}_R^{(2)}$	191.14	$\hat{Y}_R^{(3)}$	325.40
$\hat{Y}_R^{(4)}$	307.37	$\hat{Y}_R^{(5)}$	191.28	$\hat{Y}_R^{(6)}$	325.31
$\hat{Y}_R^{(7)}$	313.08	$\hat{Y}_R^{(8)}$	191.22	$\hat{Y}_R^{(9)}$	325.30
$\hat{Y}_R^{(10)}$	321.20	$\hat{Y}_R^{(11)}$	191.28	$\hat{Y}_R^{(12)}$	325.31
$\hat{Y}_R^{(13)}$	191.28	$\hat{Y}_R^{(14)}$	191.27	$\hat{Y}_R^{(15)}$	325.31
$\hat{Y}_R^{(16)}$	191.11	$\hat{Y}_R^{(17)}$	100.07	$\hat{Y}_R^{(18)}$	348.99

Table 13: PRE(%) for Set 2 (case II)

Estimator	PRE(%)	Estimator	PRE(%)	Estimator	PRE(%)
$\hat{Y}_{S_R}^{(1)}$	100.00	$\hat{Y}_{S_R}^{(2)}$	58.28	$\hat{Y}_{S_R}^{(3)}$	305.40
$\hat{Y}_R$	37.15	$\hat{Y}_R^{(2)}$	183.60	$\hat{Y}_R^{(3)}$	305.40
$\hat{Y}_R^{(4)}$	286.78	$\hat{Y}_R^{(5)}$	183.72	$\hat{Y}_R^{(6)}$	305.40
$\hat{Y}_R^{(7)}$	293.74	$\hat{Y}_R^{(8)}$	183.67	$\hat{Y}_R^{(9)}$	305.40
$\hat{Y}_R^{(10)}$	301.87	$\hat{Y}_R^{(11)}$	183.72	$\hat{Y}_R^{(12)}$	305.40
$\hat{Y}_R^{(13)}$	183.72	$\hat{Y}_R^{(14)}$	183.71	$\hat{Y}_R^{(15)}$	305.40
$\hat{Y}_R^{(16)}$	183.72	$\hat{Y}_R^{(17)}$	301.87	$\hat{Y}_R^{(18)}$	330.75

7 Discussion and Conclusion

In this study, we proposed a novel Ratio-cum-Exponential in Regression-Type Mean Estimator that incorporates the Empirical Distribution Function (EDF) as a dual application of the supplementary variable within the framework of stratified random sampling. This approach was developed to address the need for more efficient estimators of the population mean by exploiting both the supplementary variable and its empirical distribution. The efficiency of the proposed estimator was rigorously analyzed through theoretical derivations, simulations, and real-world applications.

The theoretical expressions for Bias and Mean Square Error (MSE) derived up to the first-order approximation revealed the estimator's superiority over conventional estimators, including simple ratio and regression-type estimators. By integrating the EDF as supplementary information, the proposed estimator effectively reduces estimation errors, yielding the lowest MSE values compared to its counterparts. This highlights the estimator's ability to capture additional distributional information inherent in the supplementary variable, leading to significant improvements in precision.

Simulation studies and analyses based on real-world data sets further validate the proposed estimator's robustness. Across various scenarios and population structures, the estimator consistently achieved the highest Percentage Relative Efficiencies (PREs) while maintaining the lowest MSE values. This demonstrates that the Ratio-cum-Exponential in Regression-Type Mean Estimator is not only theoretically sound but also highly effective in practical applications.

The dual application of the EDF is particularly impactful in cases where the supplementary variable exhibits strong correlation with the study variable. By leveraging this correlation, the EDF provides additional insights into the population's distribution, which complements the supplementary variable and enhances the estimator's performance. This makes the proposed methodology especially valuable in stratified sampling, where population heterogeneity demands accurate and efficient estimation techniques.

The findings of this study underscore the potential of the proposed estimator to improve estimation accuracy in diverse fields, including agriculture, economics, and social sciences. The approach is particularly suitable for surveys involving stratified sampling designs, where the heterogeneity within strata can be effectively managed using the EDF. Furthermore, the estimator's ability to utilize supplementary information efficiently makes it a promising tool for tackling challenges in survey sampling, especially when traditional methods fall short.

In conclusion, the Ratio-cum-Exponential in Regression-Type Mean Estimator incorporating the EDF as dual information represents a significant advancement in the field of survey sampling. Its superior performance, as demonstrated through theoretical and empirical analyses, establishes it as a reliable and efficient estimator for population mean estimation. This study provides a strong foundation for its adoption in practical applications and offers new avenues for future research.

Future studies may explore the extension of this methodology to other sampling designs, such as systematic and cluster sampling, as well as its application in multivariate scenarios. Additionally, the estimator's performance under non-response and missing data conditions could be investigated to further enhance its applicability. These extensions would not only broaden the scope of the proposed approach but also strengthen its relevance in addressing emerging challenges in survey sampling and data analysis.

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